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We study the quantization of a model proposed by Newton to explain centripetal force namely, that of a particle moving on a regular polygon. The exact eigenvalues and eigenfunctions are obtained. The correspondence with Dicke-Wittke cage is established. The quantum mechanics of a particle moving on a circle and in an infinite potential well are derived as limiting cases.

I. INTRODUCTION

The model of a particle bouncing off a circular constraint on a polygonal path was originally devised by Newton to motivate the concept of centripetal force¹ associated with circular motion of the particle. It serves the precise purpose of highlighting the notion of a force continuously operating on the particle on a circle as the limiting case of motion on an inscribed polygon with the corners acting as force centers. However, the intuitive simplicity of the circle case has contributed to the consistent ignoring of Newton's original model. Recently the problem has received some attention^{2,3} mostly from the historical perspective. No quantum mechanical treatment of the problem is available in the literature. The quantum mechanics of the polygon model will be of much pedagogical value for introductory courses in quantum theory inasmuch as it is an illustration of the fact that the symmetry (the N -fold discrete rotational symmetry in the present case) of a problem is fully reflected in the wavefunction.

In this work we first reduce the Lagrangian for the system by using the constraint equation and then use a suitable generalized coordinate to bring the Hamiltonian to the free particle form. The resulting eigenvalues are shown to reduce to those of a particle on a circle in the limit as the number of sides $N \rightarrow \infty$ and to those of a particle in an infinite potential well in the $N = 2$ case. The correspondence with the angular momentum quantization condition for the Dicke-Wittke cage is also shown.

II. THE CLASSICAL HAMILTONIAN

We begin by constructing the full Lagrangian for a particle of unit mass constrained to move on an N -sided regular polygon. The N -gon can be parametrized by

$$r = b \sec(\xi - (2m - 1)\pi/N), \quad \frac{2(m - 1)\pi}{N} \leq \xi \leq \frac{2m\pi}{N} \quad (1)$$

where $m = 1, \dots, N$ labels the sides of the polygon. As shown in Fig. 1, $b = a \cos(\pi/N)$ is the length of the

normal from the center of the circumscribed circle (of radius a) to the side. The length of a side is then given by $c = 2a \sin(\pi/N)$.

The Lagrangian will be given by

$$L = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\xi}^2 + \lambda(r - b \sec(\xi - (2m - 1)\pi/N)) \quad (2)$$

where λ is a Lagrange multiplier implementing the constraint

$$\Omega = r - b \sec(\xi - (2m - 1)\pi/N) = 0 \quad (3)$$

To reduce the Lagrangian to the only relevant degree of freedom i.e., ξ , we employ the constraint directly to bring it to the form

$$L_r = \frac{1}{2}b^2 \sec^4(\xi - (2m - 1)\pi/N) \dot{\xi}^2 \quad (4)$$

The momentum conjugate to the coordinate ξ is

$$P_\xi = \partial L_r / \partial \dot{\xi} = b^2 \sec^4(\xi - (2m - 1)\pi/N) \dot{\xi}. \quad (5)$$

Finally, the classical reduced Hamiltonian is given by

$$H(\xi) = \frac{P_\xi^2}{2b^2} \cos^4(\xi - (2m - 1)\pi/N) \quad (6)$$

III. QUANTIZATION

A closer look at the Hamiltonian Eq. 6 reveals that if we introduce a new generalized coordinate

$$q = b \tan(\xi - (2m - 1)\pi/N) \quad (7)$$

it reduces to the simple free particle form

$$H(q) = \frac{1}{2}\dot{q}^2 \quad (8)$$

Following the usual Schrödinger prescription for quantization, the quantum canonical momentum is $P_q = -i\hbar\partial/\partial q = \hbar k$. The Hamiltonian is therefore the usual free particle Schrödinger operator

$$\mathcal{H} = -\frac{\hbar^2}{2} \frac{\partial^2}{\partial q^2}. \quad (9)$$

A more formal approach to the quantization of such a Hamiltonian Eq. 6 would be to construct the Laplace-Beltrami operator in terms of the properly constructed quantum momentum operator⁵. We emphasize that the same quantum Hamiltonian is obtained by this procedure also.

The free particle solutions are now given by the plane waves

$$\psi(q) = A e^{\pm i k q} \quad (10)$$

Reverting back to the ξ variable we have

$$\psi_m(\xi) = A e^{i k b \tan(\xi - (2m-1)\pi/N)} \quad (11)$$

for the wavefunction in the m^{th} side. The normalization is obtained from

$$\sum_{m=1}^N \int_{2(m-1)\pi/N}^{2m\pi/N} |\psi_m(\xi)|^2 d\xi = 1 \quad (12)$$

whence $A = 1/\sqrt{2\pi}$.

The boundary condition following from the singlevaluedness of the wavefunction

$$\psi_1(\xi = 0) = \psi_N(\xi = 2\pi) \quad (13)$$

leads to the quantization condition

$$\begin{aligned} k b \tan(\pi/N) &= n\pi \\ \text{or} \\ k a \sin(\pi/N) &= n\pi \end{aligned} \quad (14)$$

The energy eigenvalues are now given by

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2a^2 \sin^2(\pi/N)}. \quad (15)$$

We note that the above quantization conditions are also obtained by considering symmetric and antisymmetric solutions about the midpoint of a side, $\xi = (2m-1)\pi/N$ namely,

$$\psi_m^s(\xi) = A_s \cos(k b \tan(\xi - (2m-1)\pi/N))$$

and

$$\psi_m^a(\xi) = A_a \sin(k b \tan(\xi - (2m-1)\pi/N))$$

and imposing the periodic boundary conditions

$$\psi_m^{s,a}(\xi) = \psi_{m+1}^{s,a}(\xi + 2\pi/N).$$

In Fig. 2 we show the solutions for a hexagon ($N = 6$).

IV. THE CIRCLE ($N \rightarrow \infty$) LIMIT

The familiar example of a particle moving on a circle⁷ would correspond to the limit $N \rightarrow \infty$ for the polygon model. In this limit, as each side of the polygon reduces to just a point, it is necessary to redefine the conjugate variables as $\phi = N\xi$ for the angle variable and $l = ka/N$ for the angular momentum in units of \hbar . We see that the eigenvalues now become $E_n = n^2 \hbar^2 / 2a^2$ and the eigenfunctions reduce to the well known solutions $\psi(\phi) = A e^{\pm i n \phi}$ for a free particle on a circle.

V. THE CAGE LIMIT

In the early days of quantum mechanics, the Dicke-Wittke cage⁴ was one of the many *gedanken* experiments conceived to illustrate the consequences of the uncertainty principle. The original motivation of Dicke and Wittke was to emphasize the canonicity of the angle and the corresponding angular momentum for a cage having a large number of bars, on the basis of the uncertainty principle. As with every other such example they could estimate, based on the angle-angular momentum uncertainty relation, the order of magnitude of the angular momentum that the cage could absorb from an incident photon beam. The problem of a particle moving on a regular polygon is equivalent to the Dicke-Wittke cage when number of sides of the polygon is very large inasmuch as they share the same N -fold rotational symmetry. Indeed, the angular momentum for a particle moving freely on a polygon with large number (N) of sides is quantized in units of $N\hbar$ as is seen from the quantization condition Eq. 14 which reduces to $ka = nN$ provided one can identify ka with the angular momentum. It is to be noted that the quantization condition Eq. 14 is exact and from symmetry considerations should be valid for a cage with any number of bars.

VI. THE INFINITE POTENTIAL WELL ($N = 2$) LIMIT

In the interesting case of $N = 2$, classically the polygon reduces to just a segment of length $2a$ traversed in both directions as discussed very lucidly by Anicin². Quantum mechanically this is equivalent to a particle confined in an infinite potential well of width $2a$. Indeed, the eigenvalue equation Eq. 15 reduces to the familiar eigenvalues

$$E_n = \frac{n^2 \pi^2 \hbar^2}{8a^2}. \quad (16)$$

Note that in the above we have effected the replacement of wavevector k by $k/2$ to take care of the two-fold reflection symmetry which is inherent in the reduction of the polygon to the segment of length $2a$. We remark that since the parameterization Eq. 1 of the polygon does not

survive down to the $N = 2$ case for obvious reasons, the eigenfunctions cannot be expressed in terms of the variable q or ξ . The eigenvalue equation Eq. 15, nevertheless, is robust and gives the true eigenvalues.

VII. SUMMARY

We have quantized Newton's polygon model and derived the general eigenvalues and eigenfunctions. We have established the correspondence with the Dicke-Wittke cage and have recovered the quantum mechanics in the circle limit of the polygon. Moreover, the energy eigenvalues for the $N = 2$ case have been identified with those of a particle in an infinite potential well. In the light of the above results it would be interesting exercise for students of introductory courses on quantum mechanics to look into the polygon analogues of the well studied problems of Stark effect for rotor⁶, Aharonov-Bohm effect for a particle on a circle etc..

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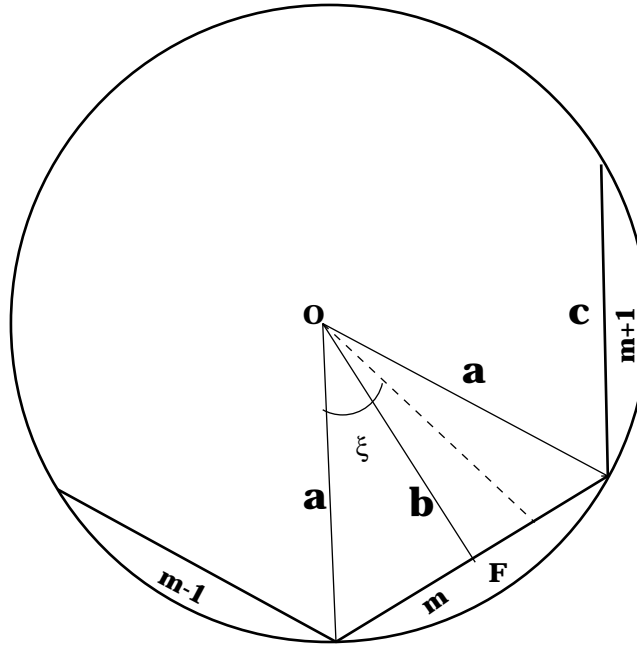


FIG. 1. The regular polygon circumscribed by a circle of radius a .

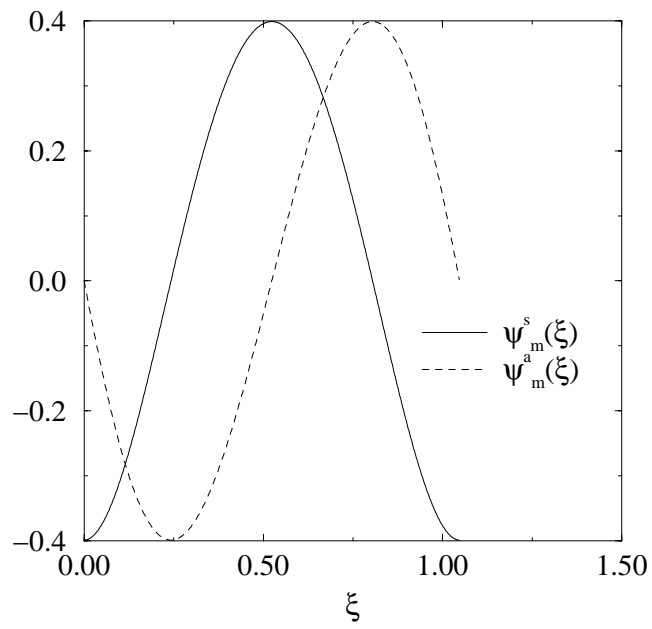


FIG. 2. Symmetric ($\psi_m^s(\xi)$) and antisymmetric ($\psi_m^a(\xi)$) ground state ($n = 1$) wavefunctions for the case of hexagon ($N = 6$).

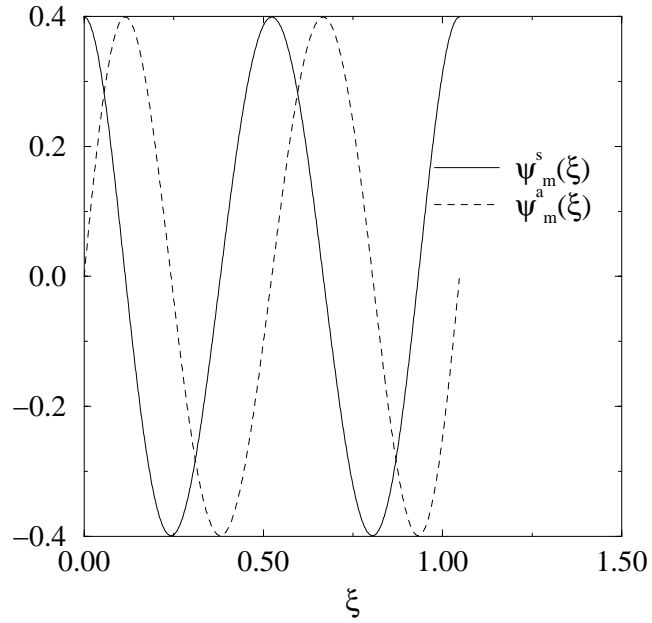


FIG. 3. Symmetric ($\psi_m^s(\xi)$) and antisymmetric ($\psi_m^a(\xi)$) first excited state ($n = 2$) wavefunctions for the case of hexagon ($N = 6$).